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Comparison Results for Related Properly and Improperly Posed Cauchy Problems for Second-Order Operator Equations

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Solutions of Cauchy problems for certain classes of second-order operator equations are compared with solutions of associated perturbed equations. Neither the original problem nor the perturbed problem is required to be well posed in the sense of Hadamard. The logarithmic convexity method is used to derive Hölder stability inequalities relating solutions of the perturbed and unperturbed problems in a suitably chosen measure. A special case is treated in order to indicate how certain data assumptions and requirements on the solutions can be relaxed as well as to demonstrate the applicability of these results to physical models.

1. INTRODUCTION

In [2], solutions of Cauchy problems for a class of first-order operator equations were compared with solutions of associated perturbed equations. It is the purpose of this paper to extend the analysis of this previous paper to Cauchy problems for certain second-order operator equations. Our goal here is to establish stability results without referring to solution representation and without requiring either the original problem or the related perturbed problem to be well posed in the sense of Hadamard.

To be more specific, we shall be interested in comparing the solution of an original problem of the form

$$\begin{gathered} Pu_{tt} + Lu_t + Mu = F(t, u, u_t), \qquad t \in [0, T), \\ u(0) = f_1, \qquad \frac{du}{dt}(0) = g_1 \end{gathered} \tag{1.1}$$

with the solution of the perturbed problem

$$\begin{gathered} Pw_{tt} + Lw_t + Mw + \varepsilon Nw = F(t, w, w_t), \qquad t \in [0, T), \\ w(0) = f_2, \qquad \frac{dw}{dt}(0) = g_2, \end{gathered} \tag{1.2}$$

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where ε denotes a small positive parameter lying in an interval $0 < \varepsilon \leqslant \varepsilon_0$. We propose to find a stabilizing constraint set such that if u and w both belong to this set and if the Cauchy data are "close" in an appropriately defined sense, then u and w will remain "close" over a finite time interval.

The logarithmic convexity method [7] will be used to compare the solutions of (1.1) and (1.2). We will show that if u and w belong to the appropriate spaces of functions, then their difference in a suitably chosen measure is of order ε to some positive power which is a function of t for $0 \leqslant t < T$.

As in [2], we do not attempt to answer the question of whether solutions of problems (1.1) and (1.2) actually exist. Since we are interested in the topic of stability, we shall assume throughout this paper that solutions do exist for the problems under consideration. We note here that if the solution w exists for a sequence of values ε_n tending to zero such that $0 < \varepsilon_n \leqslant \varepsilon_0$ and if the solution u exists, then our results indicate that w would converge to u in the chosen norm through this sequence of values as $\varepsilon_n \to 0$.

The definitions and properties of the operators and spaces involved in problems (1.1) and (1.2) will be made precise in the next section. Section 3 of this paper is then devoted to establishing the theorem which relates solutions of the problems specified in Section 2. A special case is considered in Section 4 in order to indicate how certain data assumptions and restrictions on the solutions can be relaxed. The results for this particular case are subsequently applied to a problem in the classical theory of vibrating plates in Section 5.

2. Statement of Problem

We define H to be a real Hilbert space with inner product $(\ ,\ )$ and norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$ and $D \subset H$ to be a dense linear subspace of H. Let P, L and M denote linear operators (bounded or unbounded) which map D into H and consider the problem

$$\begin{gathered} Pu_{tt} + Lu_t + Mu = F(t, u, u_t), \qquad t \in [0, T), \\ u(0) = f_1, \qquad u_t(0) = g_1. \end{gathered} \tag{2.1}$$

Here f_1 and g_1 belong to H and $T > 0$. We shall compare the solution of (2.1) with the solution of the problem

$$\begin{gathered} Pw_{tt} + Lw_t + Mw + \varepsilon Nw = F(t, w, w_t), \qquad t \in [0, T), \\ w(0) = f_2, \qquad w_t(0) = g_2. \end{gathered} \tag{2.2}$$

The initial data f_2 and g_2 belong to H, N is a linear operator mapping D into H and ε is a small positive parameter. We adopt the following hypotheses throughout this paper:

(i) The operators P, L, M and N as well as the space H are independent of t;

(ii) P is symmetric and $\exists$ a constant $\lambda > 0$ such that

$$\lambda^2(P\varphi, \varphi) \geqslant \|\varphi\|^2 \qquad \text{for all } \varphi \in D;$$

(iii) L is a symmetric, positive semi-definite operator;

(iv) M and N are symmetric;

(v) the solutions of (2.1) and (2.2) belong to $C^2([0, T); D)$;

(vi) the nonlinear term $F(t, z, z_t)$ satisfies, for $z_1, z_2 \in C^2([0, T); D)$, an inequality of the form

$$\|F(t, z_1, (z_1)_t) - F(t, z_2, (z_2)_t)\| \leqslant K_1 |(Py, y)^{1/2}| + K_2 |(Py_t, y_t)^{1/2}| + K_3 |(Ly, y)^{1/2}|,$$

where $y = z_1 - z_2$ and the K_i are nonnegative constants.

In order to compare the solutions u and w, we define $v \equiv w - u$ so that $v \in C^2([0, T); D)$ satisfies the following problem:

Problem A.

$$Pv_{tt} + Lv_t + Mv = -\varepsilon Nw + F(t, w, w_t) - F(t, u, u_t), \qquad t \in [0, T),$$
$$v(0) = f, \qquad v_t(0) = g.$$

Here the Cauchy data $f = f_2 - f_1$ and $g = g_2 - g_1$ are assumed to be small in the sense that there exist nonnegative constants k_i $(i = 1,..., 5)$ such that $(Pf, f) \leqslant k_1 \varepsilon^2$, $(Lf, f) \leqslant k_2 \varepsilon^2$, $(Pg, g) \leqslant k_3 \varepsilon^2$, $|(Mf, f)| \leqslant k_4 \varepsilon^2$ and $|(Nf, f)| \leqslant k_5 \varepsilon$.

In the next section of this paper, we will show that the solution of Problem A depends Hölder continuously on the parameter ε in an appropriate measure for $0 \leqslant t < T$.

3. Stability Results

Under the assumptions of Section 2, we now establish the following theorem:

Theorem 1. *Let u be a solution of* (2.1) *such that $\int_0^T \|Nu\|^2 \, d\eta \leqslant R_1^2$ for a prescribed constant R_1. If v is a solution of Problem* A *which lies in the* *class of functions* $\mathscr{M} = \{\varphi \in C^2([0, T); D)$: $\int_0^T \{(P\varphi, \varphi) + (T-\eta) \times (L\varphi, \varphi)\}\, d\eta \leqslant R_2^2\}$ *where* R_2 *is an a priori constant independent of* ε, *then there exist computable constants* C *and* R_3 *independent of* ε *such that on any compact subset of* $[0, T)$, v *satisfies the inequality*

$$\int_0^t [(Pv, v) + (t-\eta)(Lv, v)]\, d\eta \leqslant C\varepsilon^{2[1-\delta(t)]} R_3^{2\delta(t)}, \tag{3.1}$$

where $0 \leqslant \delta(t) < 1$.

Proof. Logarithmic convexity arguments applied to the functional

$$\begin{aligned} \phi(t) = \int_0^t [(Pv, v) + (t-\eta)(Lv, v)]\, d\eta + (T-t)(Pf, f) \\ + \tfrac{1}{2}[T^2 - t^2](Lf, f) + Q^2 \end{aligned} \tag{3.2}$$

lead to the desired result. We shall show that if the constant term Q^2 is properly chosen, then as a function of t, ϕ satisfies a second-order differential inequality of the form

$$\phi\phi'' - (\phi')^2 \geqslant -c_1\phi\phi' - c_2\phi^2 \tag{3.3}$$

for computable, nonnegative constants c_1 and c_2. As we shall see, Q^2 depends upon the initial data f and g as well as on an a priori bound on the norm of Nu.

Differentiating (3.2) we have

$$\begin{aligned} \frac{d\phi}{dt} &= (Pv, v) - (Pf, f) + \int_0^t (Lv, v)\, d\eta - t(Lf, f) \\ &= 2\int_0^t (Pv_\eta, v)\, d\eta + 2\int_0^t (t-\eta)(Lv_\eta, v)\, d\eta \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{d^2\phi}{dt^2} &= 2\int_0^t (Pv_{\eta\eta}, v)\, d\eta + 2\int_0^t (Pv_\eta, v_\eta)\, d\eta + 2\int_0^t (Lv_\eta, v)\, d\eta + 2(Pg, f) \\ &= 2\int_0^t (Pv_\eta, v_\eta)\, d\eta - 2\int_0^t (Mv, v)\, d\eta - 2\varepsilon\int_0^t (Nw, v)\, d\eta \\ &\quad + 2\int_0^t (F(\eta, w, w_\eta) - F(\eta, u, u_\eta), v)\, d\eta + 2(Pg, f). \end{aligned} \tag{3.5}$$

Expression (3.5) is obtained after substitution of the differential equation in the term $2\int_0^t (Pv_{\eta\eta}, v)\, d\eta$. Consider now the identity

$$0=\int_0^t (v_\eta, Pv_{\eta\eta}+Lv_\eta+Mv+\varepsilon Nw-[F(\eta,w,w_\eta)-F(\eta,u,u_\eta)])\,d\eta$$
$$=\tfrac{1}{2}[(Pv_t,v_t)+(Mv,v)+\varepsilon(Nv,v)]-\tfrac{1}{2}[(Pg,g)+(Mf,f)+\varepsilon(Nf,f)]$$
$$+\int_0^t (Lv_\eta,v_\eta)\,d\eta+\varepsilon\int_0^t (Nu,v_\eta)\,d\eta-\int_0^t (F(\eta,w,w_\eta)-F(\eta,u,u_\eta),v_\eta)\,d\eta.$$

Thus,

$$-(Mv,v)=(Pv_t,v_t)+\varepsilon(Nv,v)+2\int_0^t (Lv_\eta,v_\eta)\,d\eta+2\varepsilon\int_0^t (Nu,v_\eta)\,d\eta$$
$$-2\int_0^t (F(\eta,w,w_\eta)-F(\eta,u,u_\eta),v_\eta)\,d\eta-G_1, \tag{3.6}$$

where $G_1=(Pg,g)+(Mf,f)+\varepsilon(Nf,f)$. Using this result in Eq. (3.5), we see that

$$\frac{d^2\phi}{dt^2}=4\int_0^t [(Pv_\eta,v_\eta)+(t-\eta)(Lv_\eta,v_\eta)]\,d\eta$$
$$+4\varepsilon\int_0^t (t-\eta)(Nu,v_\eta)\,d\eta-2\varepsilon\int_0^t (Nu,v)\,d\eta$$
$$+2\int_0^t (F(\eta,w,w_\eta)-F(\eta,u,u_\eta),v)\,d\eta$$
$$-4\int_0^t (t-\eta)(F(\eta,w,w_\eta)-F(\eta,u,u_\eta),v_\eta)\,d\eta+G_2,$$

where $G_2=2(Pg,f)-2tG_1$. Application of Schwarz's inequality to all but the first and last terms in the previous expression yields the inequality

$$\frac{d^2\phi}{dt^2}\geqslant 4\int_0^t [(Pv_\eta,v_\eta)+(t-\eta)(Lv_\eta,v_\eta)]\,d\eta$$
$$-4\varepsilon T\left(\int_0^t \|Nu\|^2\,d\eta\right)^{1/2}\left(\int_0^t \|v_\eta\|^2\,d\eta\right)^{1/2}$$
$$-2\varepsilon\left(\int_0^t \|Nu\|^2\,d\eta\right)^{1/2}\left(\int_0^t \|v\|^2\,d\eta\right)^{1/2}$$
$$-2\int_0^t \|F(\eta,w,w_\eta)-F(\eta,u,u_\eta)\|\,\|v\|\,d\eta$$
$$-4\int_0^t (t-\eta)\,\|F(\eta,w,w_\eta)-F(\eta,u,u_\eta)\|\,\|v_\eta\|\,d\eta+G_2. \tag{3.7}$$

Consider now the integral $J_1 = -4\int_0^t (t-\eta)\,\|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\| \times \|v_\eta\|\, d\eta$. To bound this integral, we make use of the arithmetic–geometric mean inequality and the hypotheses on the term F and the operator P. The following inequalities can be obtained:

$$\begin{aligned}
J_1 \geqslant &- 4\lambda \int_0^t (t-\eta)\,|(Pv_\eta, v_\eta)^{1/2}|\,\{K_1\,|(Pv, v)^{1/2}| + K_2\,|(Pv_\eta, v_\eta)^{1/2}| \\
&+ K_3\,|(Lv, v)^{1/2}|\}\, d\eta \\
\geqslant &- \frac{2\lambda K_1}{a_1}\int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta - 2\lambda K_1 a_1 T \int_0^t (Pv, v)\, d\eta \\
&- 4\lambda K_2 \int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta \\
&- \frac{2\lambda K_3}{a_2}\int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta - 2\lambda K_3 a_2 \int_0^t (t-\eta)(Lv, v)\, d\eta \\
\geqslant &- b_1 \int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta - b_2 \int_0^t (Pv, v)\, d\eta \\
&- b_3 \int_0^t (t-\eta)(Lv, v)\, d\eta.
\end{aligned} \tag{3.8}$$

Here a_1 and a_2 are positive constants introduced by the application of the arithmetic–geometric mean inequality and the b_i $(i = 1, 2, 3)$ are positive, computable constants.

The integral $J_2 = -2\int_0^t \|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\|\,\|v\|\, d\eta$ can be bounded in a similar manner. More precisely, we have

$$\begin{aligned}
J_2 \geqslant &- 2\lambda \int_0^t |(Pv, v)^{1/2}|\,\{K_1\,|(Pv, v)^{1/2}| + K_2\,|(Pv_\eta, v_\eta)^{1/2} \\
&+ K_3\,|(Lv, v)^{1/2}|\}\, d\eta \\
\geqslant &- 2\lambda K_1 \int_0^t (Pv, v)\, d\eta - 2\lambda K_2 \left(\int_0^t (Pv, v)\, d\eta\right)^{1/2} \left(\int_0^t (Pv_\eta, v_\eta)\right)^{1/2} \\
&- \lambda K_3 a_3 \int_0^t (Pv, v)\, d\eta - \frac{\lambda K_3}{a_3}\int_0^t (Lv, v)\, d\eta,
\end{aligned} \tag{3.9}$$

where a_3 is a positive constant. In view of inequalities (3.8) and (3.9), expression (3.7) may be rewritten as

$$\begin{aligned}\frac{d^2\phi}{dt^2} \geqslant 4\int_0^t [(Pv_\eta, v_\eta) + (t-\eta)(Lv_\eta, v_\eta)]\, d\eta \\ - 4\varepsilon\lambda T\left(\int_0^t \|Nu\|^2\, d\eta\right)^{1/2}\left(\int_0^t (Pv_\eta, v_\eta)\, d\eta\right)^{1/2} \\ - 2\varepsilon\lambda\left(\int_0^t \|Nu\|^2\, d\eta\right)^{1/2}\left(\int_0^t (Pv, v)\, d\eta\right)^{1/2} - d_1\int_0^t (Pv, v)\, d\eta \\ - d_2\left(\int_0^t (Pv, v)\, d\eta\right)^{1/2}\left(\int_0^t (Pv_\eta, v_\eta)\, d\eta\right)^{1/2} - d_3\int_0^t (Lv, v)\, d\eta \\ - b_1\int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta - b_3\int_0^t (t-\eta)(Lv, v)\, d\eta + G_2 \qquad (3.10)\end{aligned}$$

for computable, positive constants d_i $(i = 1, 2, 3)$.

We shall now proceed by showing that the term $J_3 = \int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta$ in the previous inequality is bounded from above by an expression of the form $\alpha_1\phi' + \alpha_2\phi + \alpha_3 I$ where I involves initial data and the norm of Nu. Equation (3.4) may be rewritten as

$$\begin{aligned}\phi' &= 2\int_0^t\int_0^\eta \frac{d}{d\sigma}(Pv_\sigma, v)\, d\sigma\, d\eta + 2\int_0^t (Pv_\sigma, v)|_{\sigma=0}\, d\eta \\ &\quad + 2\int_0^t (t-\eta)(Lv_\eta, v)\, d\eta \\ &= 2\int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta + 2\int_0^t (t-\eta)(Pv_{\eta\eta}, v)\, d\eta \\ &\quad + 2\int_0^t (t-\eta)(Lv_\eta, v)\, d\eta + G_3,\end{aligned}$$

where $G_3 = 2t(Pg, f)$. Then, after substitution of the differential equation, it follows that

$$\begin{aligned}2\int_0^t (t-\eta)(Pv_\eta, v_\eta)\, d\eta &= \phi' + 2\int_0^t (t-\eta)(Mv, v)\, d\eta + 2\varepsilon\int_0^t (t-\eta)(Nw, v)\, d\eta \\ &\quad - 2\int_0^t (t-\eta)(F(\eta, w, w_\eta) - F(\eta, u, u_\eta), v)\, d\eta - G_3.\end{aligned}$$

The result of (3.6) leads to the equality

$$4\int_0^t (t-\eta)(Pv_\eta, v_\eta)\,d\eta = \phi' - 2\int_0^t (t-\eta)^2 (Lv_\eta, v_\eta)\,d\eta$$
$$- 2\varepsilon\int_0^t (t-\eta)^2 (Nu, v_\eta)\,d\eta + 2\varepsilon\int_0^t (t-\eta)(Nu, v)\,d\eta$$
$$+ 2\int_0^t (t-\eta)^2 (F(\eta, w, w_\eta) - F(\eta, u, u_\eta), v_\eta)\,d\eta$$
$$- 2\int_0^t (t-\eta)(F(\eta, w, w_\eta) - F(\eta, u, u_\eta), v)\,d\eta + G_4.$$

Here $G_4 = G_1 t^2 - G_3$. Application of the Schwarz and arithmetic–geometric mean inequalities as well as reference to the hypothesis on the operator P yield

$$4\int_0^t (t-\eta)(Pv_\eta, v_\eta)\,d\eta$$
$$\leqslant \phi' + a_3\lambda^2 \int_0^t (t-\eta)^2 (Pv_\eta, v_\eta)\,d\eta + \frac{\varepsilon^2}{a_3}\int_0^t (t-\eta)^2 \|Nu\|^2\,d\eta$$
$$+ \frac{\varepsilon^2}{a_4}\int_0^t (t-\eta)\|Nu\|^2\,d\eta + a_4\lambda^2 \int_0^t (t-\eta)(Pv, v)\,d\eta$$
$$+ 2\lambda\int_0^t (t-\eta)^2 \|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\|\,|(Pv_\eta, v_\eta)^{1/2}|\,d\eta$$
$$+ 2\lambda\int_0^t (t-\eta)\|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\|\,|(Pv, v)^{1/2}|\,d\eta + \bar{G}_4, \tag{3.11}$$

where $\bar{G}_4 = G_1 T^2 - 2T(Pg, f)$ and a_3, a_4 are positive constants. The last two integrals in (3.11) can be shown to satisfy the following inequalities:

$$2\lambda\int_0^t (t-\eta)^2 \|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\|\,|(Pv_\eta, v_\eta)^{1/2}|\,d\eta$$
$$\leqslant \frac{\lambda K_1 T^2}{a_5}\int_0^t (Pv, v)\,d\eta + \lambda(K_1 a_5 + 2K_2 + K_3 a_6)$$
$$\times \int_0^t (t-\eta)^2 (Pv_\eta, v_\eta)\,d\eta + \frac{\lambda K_3 T^2}{a_6}\int_0^t (Lv, v)\,d\eta, \tag{3.12}$$

$$2\lambda \int_0^t (t-\eta) \|F(\eta, w, w_\eta) - F(\eta, u, u_\eta)\| \, |(Pv, v)^{1/2}| \, d\eta$$

$$\leqslant \lambda \left(2K_1 T + \frac{K_2}{a_7} + \frac{K_3 T}{a_8}\right) \int_0^t (Pv, v) \, d\eta$$

$$+ \lambda K_2 a_7 \int_0^t (t-\eta)^2 (Pv_\eta, v_\eta) \, d\eta + \lambda K_3 a_8 T \int_0^t (Lv, v) \, d\eta. \quad (3.13)$$

Expressions (3.12) and (3.13) permit us to rewrite inequality (3.11) as

$$4 \int_0^t (t-\eta)(Pv_\eta, v_\eta) \, d\eta \leqslant \phi' + \gamma_0 \varepsilon^2 \int_0^t \|Nu\|^2 \, d\eta + \gamma_1 \int_0^t (Pv, v) \, d\eta$$

$$+ \gamma_2 \int_0^t (Lv, v) \, d\eta + \gamma_3 \int_0^t (t-\eta)^2 (Pv_\eta, v_\eta) \, d\eta + \bar{G}_4. \quad (3.14)$$

The γ_i $(i = 0, 1, 2, 3)$ are positive constants which depend on the quantities λ, K_1, K_2, K_3, T and the constants $a_j (j = 3, \ldots, 8)$ arising from application of the arithmetic–geometric mean inequality. If we let $H(t) = \int_0^t (t-\eta)^2 (Pv_\eta, v_\eta) \, d\eta$, then $H'(t) = 2 \int_0^t (t-\eta)(Pv_\eta, v_\eta) \, d\eta$ and (3.14) becomes

$$2H'(t) \leqslant \phi' + \gamma_0 \varepsilon^2 \int_0^t \|Nu\|^2 \, d\eta + \gamma_1 \int_0^t (Pv, v) \, d\eta$$

$$+ \gamma_2 \int_0^t (Lv, v) \, d\eta + \bar{G}_4 + \gamma_3 H(t). \quad (3.15)$$

The above differential inequality can be written as

$$\frac{d}{dt} [2H(t) e^{-(\gamma_3 t/2)}] \leqslant e^{-(\gamma_3 t/2)} \Big[\phi' + \gamma_0 \varepsilon^2 \int_0^t \|Nu\|^2 \, d\eta$$

$$+ \gamma_1 \int_0^t (Pv, v) \, d\eta + \gamma_2 \int_0^t (Lv, v) \, d\eta + \bar{G}_4\Big]. \quad (3.16)$$

We now assume that u belongs to that class of functions satisfying $\int_0^T \|Nu\|^2 \, d\eta \leqslant R_1^2$ for a prescribed constant R_1. With this requirement, integration of (3.16) from 0 to t leads to an inequality of the form

$$H(t) \leqslant \frac{1}{2} e^{(\gamma_3 (t-\xi)/2)} \phi(t) + \frac{1}{\gamma_3} (e^{(\gamma_3 t/2)} - 1)(\gamma_0 \varepsilon^2 R_1^2 + \bar{G}_4)$$

$$+ \frac{\gamma_1}{\gamma_3} \int_0^t (e^{(\gamma_3 (t-\eta)/2)} - 1)(Pv, v) \, d\eta + \frac{\gamma_2}{\gamma_3} \int_0^t (e^{(\gamma_3 (t-\eta)/2)} - 1)(Lv, v) \, d\eta,$$

where $0 \leqslant \xi \leqslant t < T$. Thus, there exist positive constants δ_k $(k = 0, 1, 2, 3)$ such that

$$H(t) \leqslant \delta_0 \phi + \delta_1 [\gamma_0 \varepsilon^2 R_1^2 + \bar{G}_4] + \delta_2 \int_0^t (Pv, v)\, d\eta + \delta_3 \int_0^t (Lv, v)\, d\eta. \tag{3.17}$$

Recalling that $\int_0^t (Pv, v)\, d\eta \leqslant \phi$ and that $\int_0^t (Lv, v)\, d\eta \leqslant \phi' + (Pf, f) + t(Lf, f)$, we obtain from (3.17) the result

$$H(t) \leqslant \bar{\delta}_0 \phi + \delta_1 [\gamma_0 \varepsilon^2 R_1^2 + \bar{G}_4] + \delta_3 [\phi' + (Pf, f) + T(Lf, f)].$$

Hence, it follows from (3.15) that

$$\begin{aligned} 2H'(t) &= 4 \int_0^t (t - \eta)(Pv_\eta, v_\eta)\, d\eta \\ &\leqslant \alpha_1 \phi' + \alpha_2 \phi + \alpha_3 [\gamma_0 \varepsilon^2 R_1^2 + \bar{G}_4 + (Pf, f) + T(Lf, f)], \end{aligned}$$

where α_1, α_2 and α_3 are positive constants. This inequality and the constraint on the solution u allow us to rewrite inequality (3.10) as

$$\begin{aligned} \frac{d^2\phi}{dt^2} \geqslant{} & 4 \int_0^t [(Pv_\eta, v_\eta) + (t - \eta)(Lv_\eta, v_\eta)]\, d\eta \\ & - 4\varepsilon\lambda R_1 T \left(\int_0^t (Pv_\eta, v_\eta)\, d\eta \right)^{1/2} - 2\varepsilon\lambda R_1 \left(\int_0^t (Pv, v)\, d\eta \right)^{1/2} \\ & - d_2 \left(\int_0^t (Pv, v)\, d\eta \right)^{1/2} \left(\int_0^t (Pv_\eta, v_\eta)\, d\eta \right)^{1/2} - d_4 \phi - d_5 \phi' - d_6 G, \end{aligned} \tag{3.18}$$

where $G = b_0 \varepsilon^2 R_1^2 + b_1 (Pf, f) + b_2 (Lf, f) + b_3 (Pg, g) + b_4 |(Mf, f)| + b_5 \varepsilon |(Nf, f)|$. Here the b_i $(i = 0, \ldots, 5)$ are nonnegative constants.

Using expressions (3.2), (3.4) and (3.18), we form

$$\begin{aligned} \phi\phi'' - (\phi')^2 \geqslant{} & 4S^2 + 4Q_1^2 \int_0^t [(Pv_\eta, v_\eta) + (t - \eta)(Lv_\eta, v_\eta)]\, d\eta \\ & - 4\varepsilon\lambda R_1 T \phi \left(\int_0^t (Pv_\eta, v_\eta)\, d\eta \right)^{1/2} - 2\varepsilon\lambda R_1 \phi \left(\int_0^t (Pv, v)\, d\eta \right)^{1/2} \\ & - d_2 \phi \left(\int_0^t (Pv_\eta, v_\eta)\, d\eta \right)^{1/2} \left(\int_0^t (Pv, v)\, d\eta \right)^{1/2} - d_4 \phi^2 - d_5 \phi\phi' - d_6 \phi G \end{aligned}$$

with $Q_1^2 = (T-t)(Pf, f) + \frac{1}{2}(T^2 - t^2)(Lf, f) + Q^2$ and

$$S^2 = \left\{\int_0^t [(Pv_\eta, v_\eta) + (t-\eta)(Lv_\eta, v_\eta)]\, d\eta\right\}$$
$$\times \left\{\int_0^t [(Pv, v) + (t-\eta)(Lv, v)]\, d\eta\right\}$$
$$- \left\{\int_0^t [(Pv_\eta, v) + (t-\eta)(Lv_\eta, v)]\, d\eta\right\}^2.$$

We note that both S^2 and S are nonnegative as a result of Schwarz's inequality. The term $D \equiv -d_2\phi(\int_0^t (Pv_\eta, v_\eta)\, d\eta)^{1/2}(\int_0^t (Pv, v)\, d\eta)^{1/2}$ can now be bounded in the following way:

$$D \geqslant -d_2\phi\{S^2 + (\phi')^2\}^{1/2} \geqslant -d_2\phi\{S + |\phi'|\}$$
$$\geqslant -d_2\phi\{S + \phi' + 2(Pf, f) + 2T(Lf, f)\}.$$

If we complete the square on $4Q_1^2 \int_0^t (Pv_\eta, v_\eta)\, d\eta - (4\varepsilon\lambda R_1 T\phi) \times (\int_0^t (Pv_\eta, v_\eta)\, d\eta)^{1/2}$, discard its nonnegative part as well as the nonnegative term $4Q_1^2 \int_0^t (t-\eta)(Lv_\eta, v_\eta)\, d\eta$ and apply the arithmetic–geometric mean inequality to appropriate terms, we obtain the inequality

$$\phi\phi'' - (\phi')^2 \geqslant 4S^2 - d_2\phi S - \frac{\varepsilon^2 T^2 \lambda^2 R_1^2}{Q_1^2}\phi^2 - \frac{\lambda\phi}{\sqrt{\beta_1}}\left\{\int_0^t (Pv, v)\, d\eta + \beta_1\varepsilon^2 R_1^2\right\}$$
$$- d_4\phi^2 - (d_2 + d_5)\phi\phi' - \phi[d_6 G + 2d_2(Pf, f) + 2d_2 T(Lf, f)].$$

The choice

$$Q^2 = \beta_1\varepsilon^2 R_1^2 + \beta_2(Pf, f) + \beta_3(Lf, f) + \beta_4(Pg, g)$$
$$+ \beta_5 |(Mf, f)| + \beta_6\varepsilon |(Nf, f)| \tag{3.19}$$

for some positive computable constants β_i ($i = 1, \ldots, 6$) permits us to write

$$\phi\phi'' - (\phi')^2 \geqslant 4S^2 - d_2\phi S - c_0\phi^2 - c_1\phi\phi'$$

which upon a completion of squares leads to

$$\phi\phi'' - (\phi')^2 \geqslant -c_1\phi\phi' - c_2\phi^2 \tag{3.20}$$

for computable nonnegative c_1 and c_2. As a consequence of (3.20) and the fact that $\phi(t) > 0$ for all $t \in [0, T)$, the functional $\phi(t)$ defined by (3.2) and (3.19) satisfies (see Levine [5])

$$\phi(t) \leqslant e^{-(c_2 t/c_1)}[\phi(0)]^{1-\delta(t)}[\phi(T)e^{(c_2 T/c_1)}]^{\delta(t)}, \tag{3.21}$$

where $\delta(t) = (1 - e^{-c_1 t})/(1 - e^{-c_2 T})$. In view of the bounds on the initial data, we see that $\phi(0) = T(Pf, f) + \frac{1}{2}T^2(Lf, f) + Q^2$ is $O(\varepsilon^2)$. However, if $\phi(0)$ is small, it does not necessarily follow that the product $[\phi(0)]^{1-\delta(t)}[\phi(T)e^{(c_2 T/c_1)}]^{\delta(t)}$ will be small for all $t \in [0, T)$. As noted in many prvious papers (e.g., Pucci [8], John [3]), the class of admissible solutions $v(t)$ must be restricted in order for inequality (3.21) to yield stability results. The appropriate stabilizing class is clearly indicated in (3.21), i.e., that class of functions for which $\phi(T)$ is bounded. Hence, if we assume that $v(t)$ belongs to $\mathscr{M} = \{\varphi \in C^2([0, T), X)\colon \int_0^T [(P\varphi, \varphi) + (T - \eta)(L\varphi, \varphi)]\, d\eta \leqslant R_2^2\}$ where R_2 is an a priori constant independent of ε, it follows that there exists a constant R_3 such that $\phi(T)e^{(c_2 T/c_1)} \leqslant R_3^2$. The assertion of the theorem then follows immediately from (3.21).

4. A Special Case

We consider the particular case of problems (2.1) and (2.2) in which $P = I$ (the identity operator), $L = 0$, $N = M^2$ and $F = 0$. In addition, we assume that the operator M is negative semi-definite. With such definitions of the operators, the restriction on the solution u in Theorem 1 can be relaxed. More specifically, we can establish this theorem under the assumption that u belongs to the class $\mathscr{N} = \{\varphi \in C^2([0, T); X)\colon \int_0^{t_1} \|\varphi\|^2\, d\eta \leqslant R^2\}$ for some $t_1 > T$ and a prescribed constant R.

Corollary 1. *Let $P = I$, $L = 0$, $N = M^2$ and $F = 0$ in problems (2.1) and (2.2) and let M be a negative semi-definite operator. If $u \in \mathscr{N}$ and if w satisfies $\int_0^T \|w\|^2\, d\eta \leqslant R_0^2$ for a prescribed constant R_0, then on compact subintervals of $[0, T)$,*

$$\int_0^t \|v\|^2\, d\eta \leqslant \tilde{C}\varepsilon^{2(1 - t/T)}\tilde{R}^{(2t/T)}, \tag{4.1}$$

where $\tilde{C}$ and $\tilde{R}$ are constants independent of ε.

The proof of this corollary is similar to that of Theorem 1. In this case, the functional

$$\phi(t) = \int_0^t \|v\|^2\, d\eta + (T - t)\|f\|^2 + Q^2,$$

where $Q^2 = \beta_1 \varepsilon^2 R^2 + \beta_2 \|f\|^2 + \beta_3 \|g\|^2 + \beta_4 |(Mf, f)| + \beta_5 \varepsilon \|Mf\|^2$, satisfies a second-order differential inequality of the form $\phi\phi'' - (\phi')^2 \geqslant -c_2\phi^2$ for computable, positive constants c_2 and β_i $(i = 1, \ldots, 5)$. Integration of this inequality leads to

$$\phi(t) \leqslant e^{-(c_2 t^2/2)}[\phi(0)]^{(1-t/T)}[\phi(T)e^{(c_2 T^2/2)}]^{t/T}. \tag{4.2}$$

If we restrict u to belong to $\mathscr{N}$ and w to lie in that class of functions satisfying the bound $\int_0^T \|w\|^2\, d\eta \leqslant R_0^2$ and if we require the Cauchy data to be "close" in the sense that there exist constants k_i $(i = 1, 2, 3)$ such that $\|f\| \leqslant k_1 \varepsilon$, $\|g\| \leqslant k_2 \varepsilon$, and $\|Mf\| \leqslant k_3 \varepsilon$, then expression (4.2) gives us the desired continuous dependence result (4.1). We note that the restrictions on u and w imply that $\phi(T)$ is bounded while the requirements on the data assure us that $\phi(0)$ is $O(\varepsilon^2)$.

We shall indicate here how such specializations of the operators lead to a less restrictive constraint class for the solution u. If M is negative semi-definite, the introduction of a suitable cutoff function permits us to bound $\int_0^t \|M^2 u\|^2\, d\eta$ in terms of the initial data and $\int_0^{t_1} \|u\|^2\, d\eta$ for some $t_1 > T$. To see this, choose the function $\gamma(t) \in C^2$ $(t \geqslant 0)$ defined as follows:

$$\gamma(t) = 1, \qquad 0 \leqslant t \leqslant t_0 \leqslant T; \qquad 0 \leqslant \gamma(t) \leqslant 1, \qquad t_0 \leqslant t \leqslant t_1;$$
$$\gamma(t) = 0, \qquad t \geqslant t_1.$$

It follows that

$$\int_0^t \|M^2 u\|^2\, d\eta \leqslant \int_0^\infty \gamma(\eta)\, \|M^2 u\|^2\, d\eta = (M^2 u, Mu_t)|_{t=0} \tag{4.3}$$
$$- \frac{1}{2} \int_0^\infty \gamma''(\eta)(M^2 u, Mu)\, d\eta + \int_0^\infty \gamma(\eta)(M^2 u_\eta, Mu_\eta)\, d\eta.$$

The equality in (4.3) is obtained by substituting the differential equation $u_{tt} + Mu = 0$ (now assumed to hold for $0 \leqslant t < t_1$) and integrating by parts twice. The definiteness condition on M allows us to discard $\int_0^\infty \gamma(\eta)(M^2 u_\eta, Mu_\eta)\, d\eta$ from the bounding inequality since it is nonpositive. Then, setting $\hat{Q} = (M^2 f_1, Mg_1)$ and using the Schwarz and arithmetic–geometric mean inequalities in (4.3), we find that

$$\int_0^\infty \gamma(\eta)\, \|M^2 u\|^2\, d\eta \leqslant \hat{Q} + \frac{\alpha}{4} \int_0^\infty \gamma(\eta)\, \|M^2 u\|^2\, d\eta + \frac{1}{4\alpha} \int_0^{t_1} \frac{(\gamma'')^2}{\gamma} \|Mu\|^2\, d\eta \tag{4.4}$$

for a positive constant α. Inequality (4.4) may be rewritten as

$$\left(1 - \frac{\alpha}{4}\right) \int_0^\infty \gamma(\eta)\, \|M^2 u\|^2\, d\eta \leqslant \hat{Q} + \frac{1}{4\alpha} \int_0^{t_1} \frac{(\gamma'')^2}{\gamma} (M^2 u, u)\, d\eta$$

and then a second application of the two previously cited inequalities yields

$$\left(1-\frac{\alpha}{4}-\frac{\beta}{8\alpha}\right)\int_0^{\infty}\gamma(\eta)\,\|M^2u\|^2\,d\eta\leqslant\hat{Q}+\frac{1}{8\alpha\beta}\int_0^{t_1}\frac{(\gamma'')^4}{\gamma^3}\|u\|^2\,d\eta, \quad (4.5)$$

where β is a positive constant. We observe that $\gamma(t)$ must be sufficiently continuous in order to ensure that the integral on the right side of (4.5) exists at $t=t_1$. If γ is assumed to behave like $(t-t_1)^p$ as t approaches t_1, then we need to choose p so that $3p\leqslant 4(p-2)$ or $p\geqslant 8$. With such a choice, we see that $(\gamma'')^4/\gamma^3$ is bounded at $t=t_1$. Upon choosing α and β so that $1-\alpha/4-\beta/8\alpha\geqslant k>0$ (one possible choice is $\alpha=1, \beta=2$) and requiring $\hat{Q}$ to be bounded, it follows from (4.5) and (4.3) that if $\int_0^{t_1}\|u\|^2\,d\eta\leqslant R^2$ for a prescribed constant R, then $\int_0^t\|M^2u\|^2\,d\eta\leqslant R_1^2$ for a constant R_1 independent of ε.

5. Physical Example

In this section we illustrate the results of Section 4 with a specific example. We let $M=\Delta$, the Laplace operator and define D to be a bounded region in $\mathbb{R}^n$ with a boundary ∂D smooth enough to ensure the existence of various integrals which arise in our computations. In addition, we assume that u, w and Δw vanish on ∂D. The following two problems are then obtained:

Problem I:

$$\begin{aligned} u_{tt}+\Delta u=0 \quad &\text{in} \quad D\times[0,T),\\ u=0 \quad &\text{on} \quad \partial D\times[0,T),\\ u(x,0)=f_1(x), \quad u_t(x,0)=g_1(x), \quad &x\in D; \end{aligned} \quad (5.1)$$

Problem II:

$$\begin{aligned} w_{tt}+\Delta w+\varepsilon\Delta^2 w=0 \quad &\text{in} \quad D\times[0,T),\\ w=0, \quad \Delta w=0 \quad &\text{on} \quad \partial D\times[0,T),\\ w(x,0)=f_2(x), \quad w_t(x,0)=g_2(x), \quad &x\in D. \end{aligned} \quad (5.2)$$

We observe that Problem II is well posed, while Problem I, the Cauchy problem for the Laplace equation, is improperly posed. In this case, u belongs to that class of functions which are continuous in $\bar{D}\times[0,T]$, twice continuously differentiable in $[0,T)$ and $C^4(D)$. Since Problem II is properly posed, we take w to be its classical solution. The f_i and g_i $(i=1,2)$ are assumed to be sufficiently regular prescribed functions.

The comparison of Problems I and II may be viewed in two different contexts. We might be interested in approximating the solution of the improperly posed problem by that of a well-posed problem. In this case, (5.2) may be regarded as a comparison problem for (5.1) that is obtained via the quasireversibility method [4]. Alternatively, our concern might be with the behavior of the solution w as the parameter $\varepsilon \to 0$. Thus, we might ask the question of whether in some sense the solution of Problem II converges to that of the simpler Problem I as $\varepsilon \to 0$.

An important motivation for examining problems such as (5.1) and (5.2) is derived from the fact that they appear in mathematical models of physical processes. One such example occurs in linear plate theory. More specifically, let us consider the problem of the bending of a vibrating rectangular elastic plate under the action of a uniform compressive force in the middle plane of the plate. We denote by $w(x, y, t)$ the deflection of the plate away from its initial position in the region $D = \{(x, y)\colon 0 < x < a,\ 0 < y < b\}$. If we adopt the classical linear theory of vibrating elastic plates, then w will satisfy the equation

$$\rho_0 w_{tt} + k\Delta w + \hat{D}\Delta^2 w = 0, \tag{5.3}$$

where ρ_0, k and $\hat{D}$ are positive constants which denote the density of the plate (assumed to be uniform), the magnitude of the in planar forces, and the flexural rigidity of the plate, respectively. If we set $t^* = \sqrt{(k/\rho_0)}\,t$, Eq. (5.3) may be rewritten as

$$w_{tt} + \Delta w + \varepsilon\Delta^2 w = 0, \tag{5.4}$$

where $\varepsilon = \hat{D}/k$ and the *'s have been omitted. In order to complete the problem, we assume that the plate is simply supported and prescribe its initial displacement and initial velocity. Thus, we obtain a mathematical description of the form (5.2). Suppose we are interested in this problem for $\varepsilon \ll 1$ (e.g., the case of a very thin plate). One question that arises is how the solution w compares to the solution of the problem obtained by setting $\varepsilon = 0$. If we allow for small variations in the initial data, we are led to consideration of the ill-posed problem (5.1). By not limiting ourselves to absolutely precise Cauchy data, there is some hope that a "solution" of (5.1) will exist. We remark that in the present context, this problem describes the vibration of a membrane in compression.

If we assume that the solution of (5.1) does in fact exist, we can determine appropriate conditions on u and w so that $v = w - u$ satisfies a stability inequality on compact sets of $[0, T)$. Such an inequality follows directly from Corollary 1. If we adopt the $\mathscr{L}^2$ inner product and, in addition to the hypotheses of the corollary assume that the initial data for the difference problem are small in the sense indicated in Section 4 and that the data term $\int_D (\Delta^2 f_1)(\Delta g_1)\,dx\,dy$ is bounded, then we obtain the result

$$\int_0^t \int_D (w-u)^2\,dx\,dy\,d\eta = O(\varepsilon^{\delta(t)}), \tag{5.5}$$

where $\delta(t) = 2(1 - t/T)$ for $0 \leqslant t < T$. Thus, if ε is sufficiently small, (5.5) indicates that u will be arbitrarily close to w in the given measure on $[0, T)$.

Remark 5.1. In the event that either u or u_t is identically zero on $t = 0$, the term $\int_D (\Delta^2 f_1)(\Delta g_1)\,dx\,dy$ vanishes and, consequently, the boundedness restriction on the initial data is unnecessary. Motivated by this observation, we could decompose (5.1) into two problems, each of which has a homogeneous piece of initial data. The solution to each of these problems could then be compared with the solution of the corresponding well-posed problem which is obtained from a similar decomposition of (5.2). If we handle Problem I in this way, we find that not only the indicated data restriction is circumvented but also solutions of the "simpler" problems are more likely to exist.

6. Concluding Remarks

It is appropriate here to point out that, in addition to problem (2.2), there are other perturbations of (2.1) which lead to the same type of results presented in Section 3. In particular, we can obtain Hölder stability inequalities using a perturbed problem of the form

$$Pw_{tt} + Lw_t + Mw + \varepsilon\tilde{N}w_{tt} = F(t, w, w_t), \qquad t \in [0, T),$$
$$w(0) = f_2, \qquad \frac{dw}{dt}(0) = g_2, \tag{6.1}$$

where $\tilde{N}$ is a positive semi-definite symmetric linear operator. These inequalities can be obtained by applying logarithmic convexity arguments to the functional

$$\phi(t) = \int_0^t \{(Pv, v) + (t-\eta)(Lv, v) + \varepsilon(\tilde{N}v, v)\}\,d\eta$$
$$+ (T-t)\{(Pf, f) + \varepsilon(\tilde{N}f, f)\} + \frac{1}{2}(T^2 - t^2)(Lf, f) + Q^2.$$

Here $v = w - u$ where w and u are the solutions of (6.1) and (2.1), respectively. If the constant term Q^2 is properly chosen, it can be shown that $\phi(t)$

satisfies a second-order differential inequality of the form (3.3). As in the case of (3.2), Q^2 will depend on the initial data as well as a priori bounds on a functional of the solution u. Hölder continuous dependence can then be deduced once we restrict the solutions u and w to belong to the appropriate classes of functions.

Throughout this paper, we have assumed that the operators as well as the spaces are independent of t. However, under certain assumptions (see Levine [6]), we could treat problems having operators which depend on t. One topic to be pursued is the extent to which we can generalize the operators in the equations considered here as well as the class of equations itself and still guarantee Hölder stability.

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